Subtraction Strategies from Children's Thinking: Moving toward Fluency with Greater Numbers

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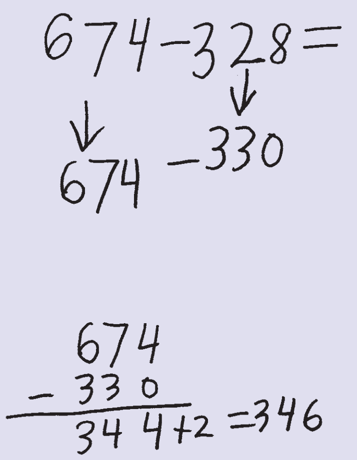
Richfield Elementary School is in the midst of its literacy campaign. Room 24 has read 674 books and Room 16 has read 328 books. The teacher, Meghan Steinmeyer, has asked her students in Room 16 to figure out the number of books they must read to catch up to Room 24. One student, John, works with the number 330 rather than 328 to make the subtraction easier (see **fig. 1**). He subtracts 330 from 674 and gets 344. Because he subtracted a number that he increased by 2, he compensates by adding 2 back to the 344 to get 346.

Learning computation is an active problem-solving process and a collaborative one in this third-grade classroom. The students entered the classroom with some strategies for subtracting lesser numbers and now need to extend these methods and move toward fluency with greater numbers. Fluency means that students can flexibly choose computational strategies to solve problems, understand and explain their methods, and produce accurate answers efficiently (NCTM 2000; Russell 2000).

The purpose of this article is to share what we have learned about supporting students as they move toward fluency with subtracting greater numbers. We believe that meaningful computation occurs at the intersection of number relationships, understanding of operations, and children's ways of thinking. We begin with examples of children's strategies that highlight relationships among numbers and operations. Then we examine whole-class discourse that promotes understanding of strategies and efficiency. Finally, we present a collection of common strategies and consider the connection between a teacher's own fluency and teaching for computational fluency.

FIGURE 1

John solved 674 – 328 by subtracting a "nice," or easier, number to work with and then compensated for the change.



Computational Strategies with a Focus on Number and Operational Relationships

In the past, we expected our students to recall and use a procedure that they had been taught for subtraction. We now ask them to think first about number and operation relationships and what might be good ways to approach working with the particular numbers. This approach keeps the emphasis on working with quantities rather than individual digits. Consider the following problem: 834 – 398. Children accustomed to letting number relationships guide them are more likely to reason 834 – 400 = 434, then add 2 to the answer to get a difference of 436.

When children are comfortable decomposing or breaking apart numbers and thinking

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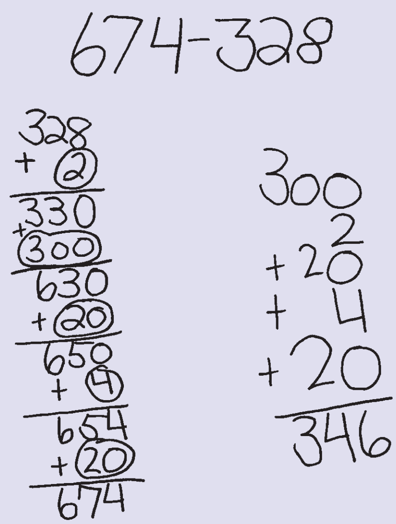
about them in many ways, they naturally use this knowledge as the basis for computation. The repertoire of strategies that students develop is derived from looking at the numbers first, identifying relationships, then playing with those relationships (Fosnot and Dolk 2001). We want students to become accustomed to asking themselves, "What number relationships can I use to solve this problem? What do I know about the relationships among the operations that I can use to approach this situation?"

In our third-grade classroom, Jamese started at 328 and added up in jumps and leaps to get to 674 (see **fig. 2**), using her understanding that subtraction is the inverse of addition. Jamese first added 2 to get to the nearest multiple of ten: 328 + 2 = 330. Next, she made a leap by adding 300 "because I can get to 674 faster." She continued, "And that equals 630. So I added 20 more and I got 650. Then I added 4 more and I got 654, so I still had to add 20 to get 674. And then I took the 300, the 2, the 20, the 4, and the 20 and added them."

FIGURE 2

Jamese solved 674 – 328 by beginning with 328 and adding up in jumps and leaps until she reached 674.



Jamese's strategy clearly is grounded in her understanding of place value; she works flexibly with hundreds, tens, and ones to add up. She shows confidence in her strategy, but she should be encouraged to consider ways to add up in fewer steps. For example, another student reasoned 328 + 72 = 400, using his knowledge of hundred pairs. Next, working with hundreds, he added 400 + 200 = 600, then made the final jump, 600 + 74 = 674. He then added 72 + 200 + 74 = 200 + 146 = 346. As students discuss and compare variations of this adding-up strategy, they become more capable of taking bigger jumps and computing more efficiently.

As Meghan reflected on her students' use of the adding up-strategy, she commented, "I noticed right away that my students like to add more than subtract and that they do not like to deal with problems where they would traditionally have to regroup. We have spent lots of time examining how the operations are closely related, and the most popular strategy probably is adding up. The students like to get to those friendly numbers ending in 0 and add up from there by 10, 20, 50, 100, or a combination of those numbers."

Orchestrating Whole-Class Discourse of Strategies

We have been carefully considering how to use whole-class discussion of strategies effectively to move students toward fluency. Initially, we simply had students present their strategies to one another, but this was not much more than "show and tell." Although the other children were attentive, the focus was not on furthering the learning of all students. We now use these discussions as "occasions for students to build understanding, learn new strategies, and reflect on the ideas of their classmates" (Trafton and Midgett 2001, p. 535). As students explain their strategies, the teacher ensures that they highlight critical steps. The following discussion excerpts occurred as Keisha explained her strategy for solving 674 – 328 (see **fig. 3**).

Teacher. Tell us what you did.

Keisha. I broke this number apart [pointing to 328].

Teacher. How did you break it apart?

Keisha. This number [3] stands for 300. This number [2] stands for 20. And this one [8] stands for 8.

The teacher made sure the other students understood where Keisha got the numbers that she was subtracting. Keisha made a "think cloud" showing how she broke apart 328 (see **fig. 3**). We have deliberated how much students should record or write down. Do we really want them to record every step? Isn't it OK for them to hold some of this knowledge in their heads? We have found that think clouds encourage students to note some of the important number relationships they used without bogging down their thinking process. Students often go back and add these clouds as they prepare to present their strategies to the class. The think clouds enable them to highlight important reasoning and invite other students to think along with them.

Teacher. Then what did you do?

Keisha. And then I subtracted 300, and that equals 374. Then I subtracted 20 from that number, and that equals 354. Then I subtracted 4 instead of 8.

Teacher. Why did you subtract 4 instead of 8?

Keisha. Because it would be too hard to count

back from 8. So I subtracted 4 and that equals 350.

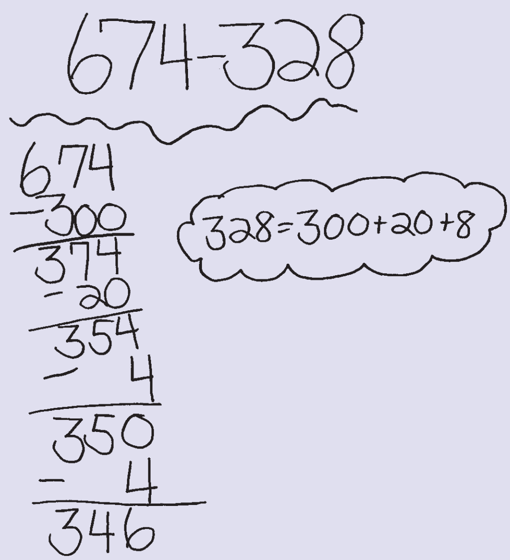
Teacher. Were you done then?

Keisha. No. I only subtracted 4, but I was supposed to subtract 8. So then I had to subtract another 4 from 350 and that equals 346.

Discourse on a particular strategy supports the individual student's reasoning and makes that reasoning available to other students for inspection.

FIGURE 3

Keisha solved 674 – 328 by breaking apart 328 into 300 + 20 + 4 + 4, then subtracting each part separately.



Some students wondered where Keisha got the 4s, so the teacher prompted her to highlight this reasoning. "When we work on subtraction problems, my students often break apart the number and subtract parts at a time," Meghan explained. "My students are very comfortable decomposing numbers and using friendly numbers from our work with writing equations for the number of the day. I was glad to see Keisha look at 354 and 8 and construct an easy way to work with these numbers. She broke the 8 apart and subtracted one 4 and then the other 4. Flexibly working with numbers is a goal for Keisha."

As students present their strategies, we ask ourselves three questions: Is the student able to clearly explain the strategy or does the student get lost or confused in the reasoning? Does the student have a clear and organized way to record the strategy? Could the strategy be refined to be more efficient? Accuracy is ensured when students have a clear way to record their strategies. Students often know what is going on in their heads but need support to get it on paper. Students' written work should clearly show how they reasoned and should be mathematically correct. Teachers also must consider whether the strategy is efficient enough to be used regularly with greater numbers. It is not efficient if students must count tallies, count on or back by ones or by small jumps, or spend an especially long time working through a procedure.

Students learn from one another as they analyze and discuss strategies. They may learn a new strategy or find ways to use a particular strategy more efficiently. We know that students will not immediately pick up a new strategy, but each time a strategy is discussed children gain additional insights through other children's explanations. A child may begin to feel comfortable with a particular strategy and then try to use it. Eventually, the child is able to integrate the strategy into his or her repertoire and use it regularly (Trafton and Thiessen 1999).

Teacher. Do you always start with hundreds?

Keisha. When there's three numbers I do. It just seems to make more sense that way.

Teacher. What if the number didn't have three numbers? What would you do then?

Keisha. Well, if there are four numbers, I start with the thousands. I always start with whatever is the largest.

In the above example, the teacher wanted the students to consider the generalizability of the strategy. Does the strategy work for any numbers, or only for these particular numbers or similar numbers, such as three-digit numbers? The class discussed whether the strategy would work if they started with the ones going from right to left or with some other place value. They also discussed and compared the advantages of these variations. Like Keisha, most students preferred to start with the greatest place value. They explained that it was easier to keep track of their reasoning because they subtracted the number the same way they said it. For example, they would say the number 328 as "three hundred twenty-eight," and the order in which they would subtract it is by beginning with the "three hundred," then the "twenty," then the "eight."

Analysis of strategies through whole-class discourse is an essential component of moving students toward fluency with greater numbers. "I have learned that I must ask my students a lot of questions to get them to think more deeply about the strategies, such as 'How did you solve that problem? Why did you choose those numbers? Can you explain someone else's strategy?'" Meghan explained that she often hears her students asking one another these questions. "I have learned that communication and sharing are an essential part of my teaching. Students learn so much from each other and I learn so much about where my students are mathematically. As a teacher, I need to meet them where they are and help move them toward where they need to be—working fluently with greater numbers. This time for whole-class discourse around the strategies is so valuable; the students love it and can't wait to share, and I love to hear what they have to say."

Consolidating toward a Few Efficient Strategies

The goal is to help children develop fluency with computation, not simply to come up with a lot of strategies. *Principles and Standards for School Mathematics* states, "As students move from third to fifth grade, they should consolidate and practice a small number of computational algorithms for addition, subtraction, multiplication, and division that they understand well and can use routinely" (NCTM 2000, p. 155).

In our third-grade classroom, DeJuan compared the number of books that each class read by subtracting each place value, then combining the partial differences (see **fig. 4**). He explained, "I took the 600 and the 300 and subtracted them to get 300. I subtracted the 70 and the 20 to get 50. And I subtracted the 4 and the 8. So what I did is, I have 4 dollars and 8 dollars, and I give my 4 dollars and I still have 4 that I owe, so it's a minus 4. So what I did is, I added 300 + 50 to get 350 dollars and I subtracted 4, because it's a minus 4. I still owe 4. That's how I got 346." Although DeJuan's use of negative numbers may seem surprising, many students use them with little difficulty when inventing their own computational strategies. They may not actually think of these partial differences as negative and positive numbers but simply consider them to have a deficit quantity (Carroll and Porter 1998).

The students in this classroom are often asked to examine and compare one another's methods: "Let's start by looking closely at Keisha's strategy [see **fig. 3**] and at DeJuan's strategy [see **fig. 4**]. Talk with your partner and discuss what you notice about these strategies. Then we will share our thinking as a class." By finding relationships among strategies, the students further refine and consolidate their methods.

Teacher. Juanita, why don't you start by telling us what Erik and you were discussing?

Juanita. We noticed on both papers they broke up their numbers.

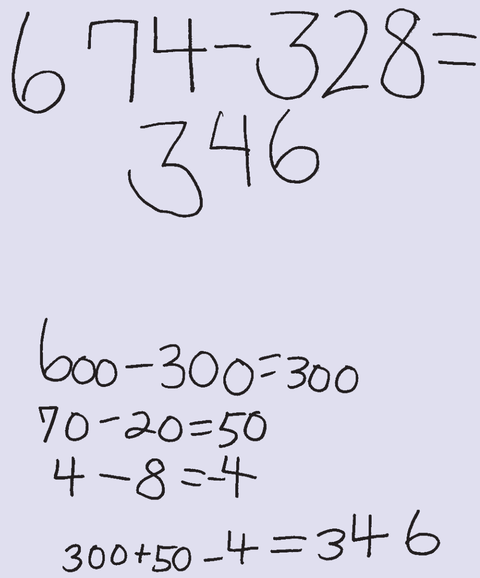
Teacher. They both broke up their numbers. Hmmm. Who can say more about how they broke up the numbers?

Germaine. Well, Keisha kept 674 the same and then broke up 328 into 300, 20, and 8. Then she subtracted the 300 and then the 20. Then she knew it was too hard to subtract 8 from 354, so she first subtracted one 4 and then the other 4.

Teacher. OK, Germaine, so you noticed that Keisha kept 674 the same as she began to subtract.

FIGURE 4

DeJuan solved 674 – 328 by subtracting each place value beginning with the greatest place, then combining the partial differences.



Let's have someone else tell us how that compares to what DeJuan did.

Houa. DeJuan broke up both numbers, and he subtracted using what he knew about the places of the numbers.

Teacher. Places of the numbers. Tarra, what does it mean that DeJuan subtracted using the places of the numbers?

Tarra. He first subtracted by starting with the hundreds place, then subtracted the numbers in the tens place, and then the numbers in the ones place.

Teacher. I wonder which strategy is faster to use. I want you to talk with your partner and discuss your thinking together about both of these strategies and which one is easier to use based on the numbers in the problem. Then I will give you a few minutes to write your thinking on paper.

Labeling strategies creates a common language in the classroom

Having students examine the strategies closely, explain verbally what they understand about the strategies, then write their thinking on paper helps them clarify and deepen their understanding of what they are learning. Meghan remarked, "It also helps me understand if my students are looking at the numbers to decide what strategy to use. I want my students to look at the numbers and from the numbers decide which strategy is most efficient to use."

Students initially experiment with several strategies and only gradually begin to match strategies to the numbers in the problem. For example, DeJuan recently started to use the "subtract each place" strategy and now uses it for most problems. Through class discourse, we want him to notice that some problems lend themselves to using numbers that are easier to work with and then compensating, and that keeping one number intact and subtracting the other number in parts is often more efficient. Students do not immediately see these connections and may not see them at all unless they are examined and discussed. The comparison of strategies brings to the forefront the efficiency of fitting the strategy to the numbers.

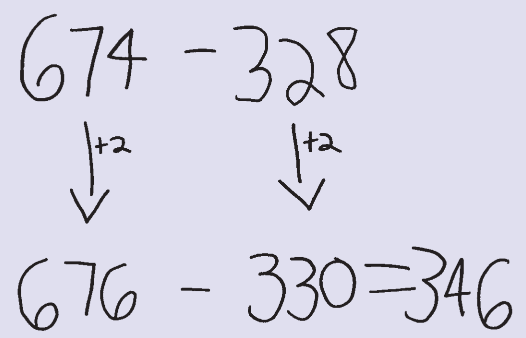
A Collection of Common Strategies from Children's Thinking

At first, it is exciting to see all the different strategies that children are capable of developing for computation. A closer examination of children's strategies, however, reveals similarities among them, and several common approaches emerge (Fosnot and Dolk 2001; Russell 1999; Trafton and Thiessen 1999). We use these common strategies as a framework for our work with children. This article has examined four approaches for subtraction (see **figs. 1–4**) that emerge from children's thinking: (1) use a number that is easier to work with and compensate, (2) add up from the subtracted number, (3) subtract the number in parts, and (4) subtract each place value.

Figure 5 shows a fifth strategy in our framework. Robert is using the "change to an easier equivalent problem" strategy to solve 674 – 328. Although this strategy for subtraction is not as common with children, it is a powerful strategy that adults use (Fosnot and Dolk 2001; Russell 1999). Robert demonstrated confidence in his use of the approach. He maintained the difference, or distance, between the numbers by adding 2 to each of the numbers and then solving 676 – 330 = 346. He explained that he had to add 2 to both numbers because "it wouldn't be fair otherwise." Most students find this strategy perplexing, so the teacher prompted Robert to give another example in an attempt to clarify how the strategy works. He explained, "It doesn't change your answer if you add 2 to both numbers. I know that 5 – 3 = 2, and then if you add 2 to 5 and 2 to 3, you have a new problem, 7 – 5 = 2, but it still gives you the same answer."

FIGURE 5

Robert solved 674 – 328 by adding 2 to each of the numbers, which maintained the difference, or distance, between them while changing it to an easier equivalent problem.



Labeling strategies creates a common language in the classroom to analyze and compare strategies. We do not label them with children's names, but rather with phrases descriptive of how they work. The labels vary from year to year because we try to draw them out from the children's language. We often post children's work on large chart paper in the room and label the strategies. As children develop in their thinking, the posters serve as an entry point for students who are struggling with ways to begin their work. The posters also act as a prompt for students to try a new approach.

Our framework includes a sixth strategy, the traditional regrouping, or borrowing, algorithm. Although this strategy does not often emerge naturally from children's thinking, it does appear in the classroom. Children may have learned it from a previous teacher or from a family member. If students choose to use the traditional algorithm, we expect them to be able to explain it and understand it. We also draw parallels to other strategies, such as subtracting each place and subtracting the number in parts. This algorithm is considered simply one more strategy in the child's repertoire.

"I find it challenging when students come to my classroom having already memorized the traditional algorithm for subtraction," Meghan reflected. "The algorithm often makes little or no sense to them, yet they think it is the only way to do math. They are afraid to let go of the rules and learn something new because they were told this was the way to follow the rule, and that is that. However, once they find a strategy that they are comfortable with, it's an entry point for them. They experience some level of success, begin to enjoy explaining how they thought about the numbers, and become more willing to try other strategies that may prove to be more efficient for them."

The Connection between Teacher Fluency and Teaching for Computational Fluency

A teacher asked us, “Why would I want to do it that way, when I can do it the old way?” She was already fluent in using a memorized rule, the traditional borrowing algorithm for subtraction, but she was not fluent in looking for and using number relationships. To her, mathematics was not about making sense by using numbers but about employing rules. Becoming familiar with the range of ways that children might work with numbers to solve subtraction problems has allowed us to work more successfully in moving students toward fluency with greater numbers.

By working toward our own fluency with each of these strategies, we have a better understanding of what to expect from our students, what significant advances in reasoning to look and listen for, and what to highlight as we analyze and compare strategies. We are more comfortable thinking about number relationships, not just rules, and we are better able to question and guide students. You may not see all these strategies emerge from your students, or you may see all of them and more. We do not directly teach children all these strategies, but as students move toward fluency with greater numbers, we are prepared to deal with the strategies that might emerge. The common strategies create a framework for us as children examine number relationships and as we share and discuss strategies. This knowledge helps each of us as we work within the context of children’s thinking.

References

Carroll, William M., and Denise Porter. “Alternative Algorithms for Whole-Number Operations.” In *The Teaching and Learning of Algorithms in School Mathematics*, edited by Lorna J. Morrow, pp. 106–14. Reston, Va.: National Council of Teachers of Mathematics, 1998.

Fosnot, Catherine Twomey, and Maarten Dolk. *Young Mathematicians at Work: Constructing Number Sense, Addition, and Subtraction*. Portsmouth, N.H.: Heinemann, 2001.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

Russell, Susan Jo. “Developing Computational Fluency with Whole Numbers.” *Teaching Children Mathematics* 7 (November 2000): 154–58.

———. *Relearning to Teach Arithmetic: Addition and Subtraction*. Parsippany, N.J.: Dale Seymour Publications, 1999.

Trafton, Paul R., and Carol Midgett. “Learning through Problems: A Powerful Approach to Teaching Mathematics.” *Teaching Children Mathematics* 7 (May 2001): 532–36.

Trafton, Paul R., and Diane Thiessen. *Learning through Problems: Number Sense and Computational Strategies*. Portsmouth, N.H.: Heinemann, 1999. ▲

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